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NORMAL MODAL LOGICS IN WHICH THE HEYTING PROPOSITIONAL CALCULUS CAN BE EMBEDDED

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INTRODUCTION

Let t(A) be the result of prefixing the necessity operator □ to every proper subformula, save conjunctions and disjunctions, of the formula A of the language of the Heyting propositional calculus H. It is well-known that H can be embedded by t in S4, i.e. A is provable in H iff t(A) is provable in S4. Esakia (1979), and also Blok (1976), have shown that S4Grz (defined below) is the maximal normal extension of S4 in which H can be embedded by t (as a matter of fact, we find in Esakia (1979) not t, but the translation which prefixes □ to every subformula; this translation is equivalent to t as far as S4 and its normal extensions are concerned).

It is not difficult to find the minimal normal modal propositional logic K4N, weaker than S4 (this logic, considered by Lemmon and Scott (1977, pp. 68-71), will be defined below), in which H can be embedded by t (cf. Došen 1981 and 1986). We may then ask whether S4Grz is also the only maximal normal extension of K4N in which we can embed H by t, i.e. whether it is true that H can be embedded by t in a normal modal propositional logic S iff S is between K4N and S4Grz. We shall show in this paper that methods of Esakia (1979) can be adapted to answer this question affirmatively.

This result depends essentially upon considering only <u>normal</u> modal propositional logics S. For nonnormal modal logics we may have minimal and maximal logics with respect to the embedding by t whose sets of theorems differ from those of K4N and S4Grz respectively (the nonmaximality of S4Grz for nonnormal modal logics was considered by Chagrov (1985)). Our result also depends upon using the translation t and not some analogous translation, which as far as S4 and its normal extensions are concerned is equivalent to t. We shall consider in this paper the difficulties which we encounter with these other translations.

The embeddings of H in K4N and related logics, which we shall consider in the next two sections, suggest that we may show H complete with respect to Kripke-style models in which the "accessibility" relation is not a quasi-ordering, but satisfies weaker conditions. After a section on these Kripke-style models, in the final section we shall make some brief comments on modal embeddings of Heyting first-order predicate logic and Heyting arithmetic, and on modal embeddings of classical logic.

MODAL LOGICS

Our basic nonmodal propositional language L will have countably many propositional variables, the binary connectives $\to, \wedge$ and $\vee$, and the unary connective $\neg$. The modal propositional language $L\Box$ will have in addition to what we have in L the unary connective $\Box$. For formulae of L or $L\Box$ we use the schematic letters $A, B, C, \dots, A_1, \dots$ As usual, $A\leftrightarrow B$ is defined as $(A\to B)\wedge(B\to A)$.

The Heyting propositional calculus in L will be denoted by H. The system K in $L\Box$ is the classical propositional calculus extended with $\Box(A\to B)\to(\Box A\to\Box B)$ and closed under the rules: *modus ponens*, substitution for propositional variables and necessitation (i.e. from A infer $\Box A$). We write $S'\subseteq S''$ when the theorems of the system S' are included among the theorems of the system S''. A system S in $L\Box$ is normal iff $K\subseteq S$ and S is closed under the rules of K.

The normal system K4N will be obtained by extending K with $\Box A\to\Box\Box A$, $\neg\Box\neg(A\to A)$ and $\Box(\Box A\vee\Box B)\to(\Box A\vee\Box B)$. It is easy to show that S4, i.e. K plus $\Box A\to\Box\Box A$ and $\Box A\to A$, properly extends K4N. The normal system S4Grz is K extended with $\Box(\Box(A\to\Box A)\to A)\to A$. It is known that S4Grz properly extends S4 (see van Benthem and Blok 1978 and Boolos 1979, Chapter 13).

The translation t is a *one-one* mapping from L into $L\Box$ such that $t(A)$ is the result of prefixing $\Box$ to every *proper* subformula of A save conjunctions and disjunctions. More precisely, t is defined as follows, via the translation s which prefixes $\Box$ to every subformula save conjunctions and disjunctions:

$s(A)=\Box A$, where A is a propositional variable,
$s(A\to B)=\Box(s(A)\to s(B))$,
$s(A\alpha B)=s(A)\alpha s(B)$, where α is $\wedge$ or $\vee$,
$s(\neg A)=\Box\neg s(A)$;

$t(A)=A$, where A is a propositional variable,
$t(A\beta B)=s(A)\beta s(B)$, where β is $\to, \wedge$ or $\vee$,
$t(\neg A)=\neg s(A)$.

We write $H\xrightarrow{t}S$ iff for every A in L we have that A is a theorem of H iff $t(A)$ is a theorem of S, i.e. H can be embedded by t in S. The following lemma asserts that K4N is the minimal normal system in which H can be embedded by t:

Lemma 1. (1) $H\xrightarrow{t}K4N$.

(2) If S is normal and $H\xrightarrow{t}S$, then $K4N\subseteq S$.

Proof. (1) Let A be a theorem of H, and let $s'(A)$ be obtained from A by prefixing $\Box$ to every subformula of A (including conjunctions and disjunctions). Then by induction on the length of proof of A in H we can show that $s'(A)$ is a theorem of K4N. To obtain that $t(A)$ is a theorem of K4N we remove superfluous necessity operators from $s'(A)$ by using the fact that $\Box(\Box B\wedge\Box C)\leftrightarrow(\Box B\wedge\Box C)$ and $\Box(\Box B\vee\Box C)\leftrightarrow(\Box B\vee\Box C)$ are theorems of K4N, and that K4N is closed under replacement of equivalents and under the rules:

$$\frac{\Box(\Box B\to\Box C)}{\Box B\to\Box C},\qquad \frac{\Box\neg\Box B}{\neg\Box B}.$$

To prove $H\xrightarrow{t}K4N$ it remains to observe that $K4N\subseteq S4$, and appeal to the well-known fact that $H\xrightarrow{t}S4$.

(2) The minimality of K4N follows from the fact that $\Box A \to \Box(\Box(\Box C \to \Box C) \to \Box A)$, $\neg\Box\neg\Box(\Box A \to \Box A)$ and $\Box(\Box(\Box C \to \Box C) \to (\Box A \vee \Box B)) \to (\Box A \vee \Box B)$, where A,B and C are propositional variables, are t-translations of theorems of H. q.e.d.

This lemma is tied up to the particular translation t and the particular primitive vocabulary we have assumed for L and L$\Box$. It is well-known that for embedding H in S4 we may also use the translation t' which prefixes $\Box$ to every <u>proper</u> subformula (including conjunctions and disjunctions). Indeed, in K4N and its normal extensions for every A in L we have that:

t(A) is provable iff t'(A) is provable
iff s(A) is provable
iff s'(A) is provable,

where s and s' are the two translations defined before Lemma 1 and in the proof of Lemma 1. To sum up, we have the following translations:

	prefixes $\Box$ to every
t	proper subformula save conjunctions and disjunctions
t'	proper subformula
s	subformula save conjunctions and disjunctions
s'	subformula

These various translations, which are not essentially different as far as K4N and its normal extensions are concerned, induce different minimal normal modal systems to replace K4N. Namely, the minimal normal modal system S such that:

$H \xrightarrow{t'} S$ is $Kt' = K + \Box A \leftrightarrow \Box\Box A,\ \neg\Box\neg(A \to A)$;

$H \xrightarrow{s} S$ is $Ks = K + \Box(\Box A \to \Box\Box A),\ \Box\neg\Box\neg(A \to A),\ \Box(\Box(\Box A \vee \Box B) \to (\Box A \vee \Box B))$;

$H \xrightarrow{s'} S$ is $Ks' = K + \Box(\Box A \leftrightarrow \Box\Box A),\ \Box\neg\Box\neg(A \to A)$.

To demonstrate this (cf. Došen 1981 and 1986) we proceed analogously to what we had for Lemma 1, save for the following. To show that if A is a theorem of H, then s'(A) is a theorem of Ks', we use the fact that Ks' is closed under the rule:

$$\frac{\Box(A_1 \leftrightarrow A_2) \qquad \Box B}{\Box B[A_1/A_2]}$$

where $B[A_1/A_2]$ is obtained from B by replacing zero or more occurrences of A_1 by A_2; we also use the fact that if A is a theorem of H, then in Ks' we can prove $\Box(s'(A) \to t'(A))$, i.e. $\Box s'(A) \to s'(A)$ (remember that H has the disjunction property, i.e. if $B \vee C$ is provable in H, then either B or C is provable in H). To show that if A is a theorem of H, then s(A) is a theorem of Ks, we use the fact that the provability of B in K4N implies the provability of $\Box B$ in Ks, and also the facts that $B \wedge C$ is provable in H iff B and C are provable in H, and that $B \vee C$ is provable in H iff B or C is provable in H. Finally, to show that if A is a theorem of H, then t'(A) is a theorem of Kt', we use the fact that the provability of A in H implies the provability of s'(A) in Kt'; this implies the provability of t'(A) in Kt' (remember again that H has the disjunction property).

With a different primitive vocabulary in L and L$\Box$ we may also end up with a minimal normal system different from K4N. For example, if we have the constant proposition $\bot$ as primitive, instead of $\neg$ (where $\neg A$ is defined as $A \to \bot$), and if $\bot$ behaves in the translations t,t',s and s' as a propositional variable, then the minimal normal system replacing K4N in Lemma 1 will be

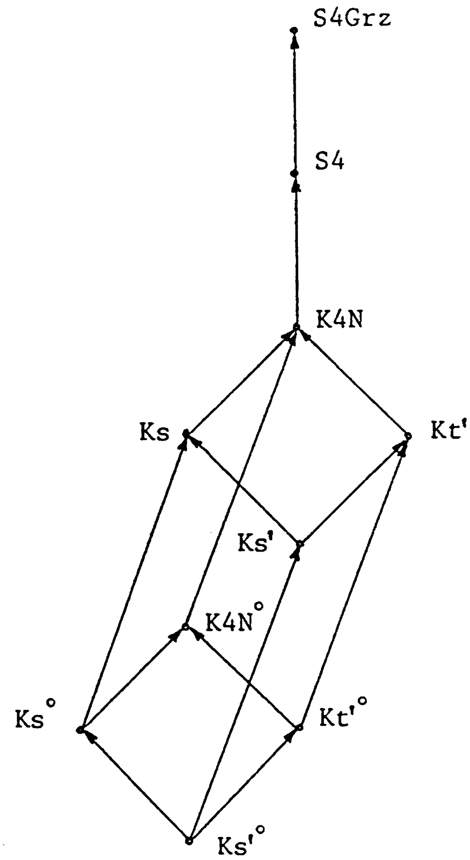

Fig. 1. Modal systems. (Arrows indicate proper inclusion.)

$K4N^\circ = K + \Box A \to \Box\Box A,\ \Box(\Box A \vee \Box B) \to (\Box A \vee \Box B)$. This difference arises because with $\bot$ primitive we have:

$$t(\neg A) = t(A \to \bot)$$
$$= s(A) \to \Box\bot,$$

whereas with $\neg$ primitive and $\bot$ defined as $\neg(A \to A)$ we have that $t(\neg A)$ is equivalent to $s(A) \to \bot$. With $\bot$ primitive, and the translations t', s and s', the minimal normal systems will be the respective systems in the list above with $\neg\Box\neg(A \to A)$ and $\Box\neg\Box\neg(A \to A)$ omitted. Let us denote these systems by Kt'°, Ks° and Ks'°. Then the modal systems which we have considered make the chart of Fig. 1.

MODAL ALGEBRAS

Let $HA = \langle H, \cap, \cup, \longrightarrow, 1, 0 \rangle$ be a Heyting algebra (called pseudo-Boolean algebra by Rasiowa and Sikorski (1963)), and let $TB = \langle B, \cap, \cup, -, 1, 0, I \rangle$ be a topological Boolean algebra (where $-$ is Boolean complement and I an interior operation). If $HA(B) = \{a \in B: Ia = a\}$, and for a and b in HA(B) we define $a \longrightarrow b$ as $I(-a \cup b)$, then $HA(TB) = \langle HA(B), \cap, \cup, \longrightarrow, 1, 0 \rangle$ is a Heyting algebra (which Esakia (1979) calls the stencil of TB).

By using a well-known construction of McKinsey and Tarski (inspired by Stone; see Rasiowa and Sikorski 1963, pp.128-130), we can embed a given Heyting algebra $HA = \langle H, \cap, \cup, \longrightarrow, 1, 0 \rangle$ in a topological Boolean algebra $TB(HA) = \langle TB(H), \cap, \cup, -, 1, 0, I \rangle$ generated by H, where for every $a \in TB(H)$ there are $b_1, \ldots, b_n, c_1, \ldots, c_n \in H$ such that $a = (-b_1 \cup c_1) \cap \ldots \cap (-b_n \cup c_n)$ and

$Ia=(b_1 \rightarrow c_1) \cap \ldots \cap (b_n \rightarrow c_n)$. It can then be shown that HA(TB(HA)) is isomorphic with HA. For every TB we can prove the following lemma:

Lemma 2. TB(HA(TB)) is isomorphic to a subalgebra of TB.

Proof. For $TB=\langle B,\cap,\cup,-,1,0,I\rangle$ let $B^*=\{a\in B: \exists b_1,\ldots,b_n,c_1,\ldots,c_n\in B$ $(b_1=Ib_1$ & ... & $c_n=Ic_n$ & $a=(-b_1\cup c_1)\cap\ldots\cap(-b_n\cup c_n))\}$. It is easy to show that $TB^*=\langle B^*,\cap,\cup,-,1,0,I\rangle$ is a subalgebra of TB. It remains to check that TB* is isomorphic with TB(HA(TB)) (a detailed proof may be found in Maksimova and Rybakov 1974, Lemmata 3.3 and 3.4). q.e.d.

So, we may always consider that TB(HA(TB)) is a subalgebra of TB, but not necessarily isomorphic with TB. If TB(HA(TB)) is isomorphic with TB, following Esakia (1979), we call TB a stenciled topological Boolean algebra.

We write TB⊨A iff for every valuation v from L□ into B we have v(A)=1 in TB, and we write TB⊨S iff for every theorem A of the system S we have TB⊨A. The essential result of Esakia which we need is the following:

Lemma 3. (Esakia 1979, Corollary 4.10) If A is not a theorem of S4Grz, then there is a finite stenciled TB such that it is not the case that TB⊨A.

We shall call $QTB=\langle B,\cap,\cup,-,1,0,I\rangle$ a quasi-topological Boolean algebra iff $\langle B,\cap,\cup,-,1,0\rangle$ is a Boolean algebra and for every $a,b\in B$ we have:

$I(a\cap b)=Ia\cap Ib$, $I1=1$, $Ia=IIa$, $I0=0$, $I(Ia\cup Ib)=Ia\cup Ib$.

Every topological Boolean algebra is quasi-topological (it satisfies moreover $Ia\leqslant a$), but not the other way round. It is easy to verify that A is a theorem of K4N iff for every QTB we have QTB⊨A; namely, the Lindenbaum algebra of K4N is a freely generated QTB.

If HA(QTB) is defined analogously to HA(TB), we can check that for every QTB the algebra HA(QTB) is a Heyting algebra (this is contained in the fact that H can be embedded in K4N by s'). For every QTB we can also prove the following analogue of Lemma 2:

Lemma 4. TB(HA(QTB)) is isomorphic to a subalgebra of QTB.

The proof of this lemma proceeds quite analogously to the proof of Lemma 2. Note that without $I0=0$ this proof might be blocked, since we might be unable to show that $0\in B^*$. And without $I(Ia\cup Ib)=Ia\cup Ib$, we might be unable to show that B* is closed under $\cup$ (and under $-$).

Note also that in QTB* (obtained as TB* in the proof of Lemma 2) we have for every $a\in B^*$ that $Ia\leqslant a$, though in QTB this is not the case for every a. Indeed, QTB*, which is isomorphic with TB(HA(QTB)), is a topological Boolean algebra. Lemma 4 yields as a corollary that for every quasi-topological Boolean algebra QTB there is a topological Boolean algebra TB which is a subalgebra of QTB and such that HA(QTB) is isomorphic with HA(TB). (Compare this with the fact, mentioned by Lemmon and Scott (1977, pp.70-71), that K4N can be axiomatized by extending K with $\Box A\rightarrow\Box\Box A$ and $\Box B\rightarrow B$, where every propositional variable of B is within the scope of a $\Box$.)

Let S be a normal system such that $K4N\subseteq S$, and let VS={QTB: QTB⊨S}. The algebras in VS make a variety (because for every theorem A of S we can ask from our QTB's that they satisfy a=1, where a is obtained from A by translating logical with algebraic symbols). Let now HA(VS)={HA(QTB): QTB∈VS}. We can prove the following (cf. Blok and Dwinger 1975, Theorem 4.1):

Lemma 5. HA(VS) is closed under homomorphic images and subalgebras.

Proof. For closure under homomorphic images, suppose QTB∈VS and f: HA(QTB)⟶HA is an onto homomorphism. By Lemma 4, we have that TB(HA(QTB)) is a subalgebra of QTB, and since VS is a variety, TB(HA(QTB))∈VS. The homomorphism f can naturally be extended to a homomorphism g from TB(HA(QTB)) onto TB(HA) (for $a=(-b_1 \cup c_1)\cap \ldots \cap(-b_n \cup c_n)$, take $g(a)=(-f(b_1)\cup f(c_1))\cap \ldots \cap (-f(b_n)\cup f(c_n))$). Since VS is a variety, TB(HA)∈VS. But HA is isomorphic with HA(TB(HA)). So, HA∈HA(VS).

For closure under subalgebras, suppose QTB∈VS and HA is a subalgebra of HA(QTB). Then TB(HA) is a subalgebra of TB(HA(QTB)). Since TB(HA(QTB)) is a subalgebra of QTB, we have that TB(HA) is a subalgebra of QTB, and since VS is a variety, TB(HA)∈VS. But HA is isomorphic with HA(TB(HA)), and, hence, HA∈HA(VS). q.e.d.

It is easy to show that HA(VS) is also closed under direct products; so, HA(VS) is in fact a variety.

By "countable" in the following two lemmata we understand "finitely or infinitely countable" (countability is assumed in these lemmata because the language L is assumed to be countable; without this assumption about L, we could prove analogous lemmata without the assumption of countability).

Lemma 6. If $H \xrightarrow{t} S$, then for every countable HA there is a QTB∈VS such that HA(QTB) is isomorphic with HA.

Proof. Suppose $H \xrightarrow{t} S$, and let s[Lind(S)]={[s(A)]: [s(A)]∈Lind(S)}, where Lind(S) is the Lindenbaum algebra of S. Of course, Lind(S)∈VS. The Heyting algebra s[S]=<s[Lind(S)],∧,∨,⟶,⊤,⊥>, where [s(A)]⟶[s(B)] is defined as □(¬[s(A)]∨[s(B)]), is a subalgebra of HA(Lind(S))∈HA(VS). So, by Lemma 5, s[S]∈HA(VS). On the other hand, s[S] can be shown isomorphic with Lind(H), the Lindenbaum algebra of H. We can define f: Lind(H)⟶s[S] by f([A])=[s(A)]. That f is a one-one mapping is shown as follows:

f([A])=f([B]) iff [s(A)]=[s(B)]
iff s(A)⟷s(B) is provable in S
iff t(A→B) and t(B→A) are provable in S
iff A⟷B is provable in H, since we have $H \xrightarrow{t} S$
iff [A]=[B].

It follows easily that f is a homomorphism and onto. So, Lind(H)∈HA(VS).

Since Lind(H) is a free Heyting algebra, there is a homomorphism from Lind(H) onto an arbitrary countable HA. According to Lemma 5, HA∈HA(VS). q.e.d.

Lemma 7. If $H \xrightarrow{t} S$, then every countable stenciled topological Boolean algebra belongs to VS.

Proof. Suppose $H \xrightarrow{t} S$, and let TB be a countable stenciled topological Boolean algebra. Then HA(TB) is a countable Heyting algebra, and by Lemma 6, there is a QTB∈VS such that HA(TB) is isomorphic with HA(QTB). Since TB is isomorphic with TB(HA(TB)), which is isomorphic with TB(HA(QTB)), we obtain by Lemma 4 that TB is a subalgebra of QTB. Since VS is a variety, TB∈VS. q.e.d.

We are now ready to prove our generalization of the theorem of Esakia and Blok:

Theorem. Let S be normal. Then $H \xrightarrow{t} S$ iff K4N⊆S⊆S4Grz.

Proof. Suppose $H \xrightarrow{t} S$. Then by Lemma 1, we have K4N⊆S. If A is not a theorem of S4Grz, then by Lemma 3 there is a finite stenciled TB such that not TB⊢A; hence, by Lemma 7, TB∈VS, and it follows that A is not a theorem of S. So, S⊆S4Grz. For the other direction it is enough to appeal to the fact that H can be embedded by t in K4N and S4Grz. q.e.d.

This method of proving our theorem depends essentially upon using the particular translation t and the particular primitive vocabulary of L and L□. To see that this is indeed the case, consider the normal modal systems from the previous section which are properly contained in K4N. These systems lack either ¬□¬(A→A) or □(□A∨□B)→(□A∨□B), and, hence, in the analogues of our quasi-topological Boolean algebras we would not have either I0=0 or I(Ia∪Ib)= Ia∪Ib. As we have remarked after Lemma 4, the lack of these principles might block our proof. So, we leave open the question what form an analogue of our theorem should take with one of the translations t', s or s', or with ⊥ primitive, instead of ¬.

KRIPKE-STYLE MODELS

The embedding of H in K4N and in weaker normal modal systems suggests that we may obtain a completeness proof for H with respect to Kripke-style models in which the "accessibility" relation is not a quasi-ordering, but satisfies weaker conditions.

Let us first consider modal Kripke models with respect to which K4N may be shown complete (cf. Lemmon and Scott 1977, pp.68-71). These models are of the form $\langle X,R,v_o\rangle$, where X is a nonempty set of "worlds" (we shall use $x,y,z,\ldots,x_1,\ldots$ as variables ranging over X), R is a binary relation over X which satisfies:

(1) $\forall x,y,z((xRy \;\&\; yRz) \Rightarrow xRz)$, i.e. R is transitive,
(2) $\forall x\exists y(xRy)$, i.e. R is serial,
(3) $\forall x,y_1,y_2((xRy_1 \;\&\; xRy_2) \Rightarrow \exists z(xRz \;\&\; zRy_1 \;\&\; zRy_2))$,

and the basic valuation v_o maps the propositional variables of L□ into PX, i.e. the power set of X. As usual, v_o is extended to a valuation v: L□⟶PX by the following recursive clauses:

$v(A)= v_o(A)$, where A is a propositional variable,
$v(A\rightarrow B)= (X-v(A))\cup v(B)$,
$v(A\wedge B)= v(A)\cap v(B)$,
$v(A\vee B)= v(A)\cup v(B)$, $v(\neg A) = X - v(A)$,
$v(\Box A)= \{x\colon \forall y(xRy \Rightarrow y\in v(A))\}$.

A formula A holds in a model $\langle X,R,v_o\rangle$ iff $v(A)= X$.

The corresponding models for H, which we shall call Ht models, are of the form $\langle X,R,v_o\rangle$, where X and R are as above, and v_o, which maps the propositional variables of L into PX, satisfies the following condition for every propositional variable A and every $x\in X$:

$$x\in v_o(A) \Leftrightarrow \forall y(xRy \Rightarrow y\in v_o(A)).$$

In ordinary Kripke models for H this condition is usually assumed only from left to right, because the converse holds trivially when R is reflexive. But in Ht models the implication from right to left is not automatically satisfied. A basic valuation v_o is extended to a valuation v: L⟶PX by the following usual recursive clauses:

$v(A)= v_o(A)$, where A is a propositional variable,
$v(A\rightarrow B)= \{x\colon \forall y(xRy \Rightarrow (y\in v(A) \Rightarrow y\in v(B)))\}$,

$v(A \wedge B) = v(A) \cap v(B)$,
$v(A \vee B) = v(A) \cup v(B)$,
$v(\neg A) = \{x: \forall y(xRy \Rightarrow y \notin v(A))\}$.

As before, A holds in an Ht model iff $v(A) = X$.

Then we can prove by induction on the complexity of A that for every formula A of L and every $x \in X$ the following holds:

(Heredity) $x \in v(A) \Longleftrightarrow \forall y(xRy \Rightarrow y \in v(A))$.

In proving Heredity, the transitivity of R is used in the cases when A is of the form $B \rightarrow C$ and $\neg B$, the seriality of R is used when A is of the form $\neg B$, whereas condition (3) for R is used when A is of the form $B \vee C$.

With the help of Heredity, we can easily verify by induction on the length of proof of A that if A is provable in H, then A holds in every Ht model. The converse, i.e. completeness, follows immediately from completeness with respect to ordinary Kripke models for H, and the fact that every ordinary Kripke model for H is an Ht model.

We may also expect to obtain models for H from models for our normal modal systems weaker than K4N, in which H can be embedded by various translations. These models for H would roughly correspond to the modal models as Ht models correspond to models for K4N. However, in these Kripke-style models for H we would have to modify in some cases the clauses for v, and in some cases the Heredity condition and the definition of holding in a model. In models which correspond to translations where $\Box$ is not omitted before disjunctions (namely, in models which correspond to Kt', Ks', Kt'° and Ks'°), the clause for $v(A \vee B)$ would be:

$v(A \vee B) = \{x: \forall y(xRy \Rightarrow (y \in v(A)$ or $y \in v(B)))\}$

rather than $v(A \vee B) = v(A) \cup v(B)$. In models which correspond to translations where $\Box$ is not prefixed only to proper subformulae (namely, in models which correspond to Ks, Ks', Ks° and Ks'°), Heredity would be replaced by the following conditional Heredity:

$\exists z(zRx) \Rightarrow (x \in v(A) \Longleftrightarrow \forall y(xRy \Rightarrow y \in v(A)))$

and holding in a model would be redefined as follows: A holds in a model iff $\forall x(\exists z(zRx) \Rightarrow x \in v(A))$. In models which correspond to systems based on $\bot$ primitive (namely, in models which correspond to the systems with the superscript $\circ$), the clause for $v(\bot)$ would be:

$v(\bot) = \{x:$ not $\exists y(xRy)\}$

rather than $v(\bot) = \emptyset$. (Since these last models need not have a serial R, we may have in them "blind worlds" in which $\bot$ holds. Heredity will guarantee that in these worlds every formula of L holds too; conditional Heredity will guarantee the same thing for blind worlds x such that $\exists z(zRx)$. This resembles the modified Kripke models of Veldman (1976) with their "exploding worlds".)

So, these various weak models for H would bring in some complications. On the other hand, for Ht models the clauses for v, as well as the Heredity condition and the definition of holding in a model, are exactly as for ordinary Kripke models for H. The only difference is in the conditions for R.

To conclude we note as a curiosity that the $\rightarrow$ and $\rightarrow, \wedge$ fragments of H can be shown sound and complete with respect to models $\langle X, R, v_\circ \rangle$ for which everything is as for Ht models save that R is only transitive. However, it would be wrong to conclude from this that these two fragments of H can be embedded by t in K4, i.e. K + $\Box A\rightarrow\Box\Box A$.

CONCLUDING COMMENTS

We shall close this paper with some brief comments on the embeddings of the Heyting first-order predicate calculus in modal first-order predicate logics, on the embeddings of Heyting arithmetic in modal extensions of Peano arithmetic, and, finally, on the embeddings of classical logic in modal extensions of Heyting's logic.

Let L_1 be the first-order language which has individual constants and variables, predicate constants, the propositional connectives of L and the quantifiers $\forall x$ and $\exists x$. The language $L_1\Box$ has $\Box$ in addition to that. First-order K in $L_1\Box$ is the classical first-order predicate calculus extended with $\Box(A\rightarrow B)\rightarrow(\Box A\rightarrow\Box B)$ and closed under: modus ponens, necessitation and universal generalization. A system in $L_1\Box$ is normal iff it includes the theorems of first-order K and is closed under its rules.

The translation t_1': $L_1\longrightarrow L_1\Box$ prefixes like t' a $\Box$ to every proper subformula of a formula A of L_1, whereas t_1: $L_1\longrightarrow L_1\Box$ prefixes $\Box$ to every proper subformula save conjunctions, disjunctions and subformulae with an initial existential quantifier. Then it is not difficult to prove that the minimal normal first-order system in which the Heyting first-order predicate calculus can be embedded by t_1' is first-order K extended with $\Box A\leftrightarrow\Box\Box A$ and $\neg\Box\neg(A\rightarrow A)$ (to prove that we use, besides the disjunction property, the analogous existence property of the Heyting predicate calculus). With t_1 instead of t_1' this minimal normal first-order system will be $K4N_1$, which is first-order K extended with $\Box A\rightarrow\Box\Box A$, $\neg\Box\neg(A\rightarrow A)$, $\Box(\Box A\vee\Box B)\rightarrow(\Box A\vee\Box B)$ and $\Box\exists x\Box A\rightarrow\exists x\Box A$. With $\bot$ primitive, instead of $\neg$, we would omit $\neg\Box\neg(A\rightarrow A)$ from these minimal systems.

With translations from L_1 into $L_1\Box$ analogous to s and s' matters are not so straightforward (as explained in Došen 1986). It is the lack of a principle like the Barcan formula which produces difficulties in finding the minimal normal first-order systems in which the Heyting first-order predicate calculus can be embedded by these translations.

Next, let us mention that first-order Heyting arithmetic can be embedded by a translation analogous to t_1 in modal extensions of first-order Peano arithmetic, with the additional operator $\Box$, which lie in between the $K4N_1$ extension of Peano arithmetic and the S4 extension of Peano arithmetic. To demonstrate that the provability of A in Heyting arithmetic implies the provability of the translation of A in $K4N_1$ Peano arithmetic, we proceed analogously to what we had for Lemma 1(1). That the provability of the translation of A in S4 Peano arithmetic implies the provability of A in Heyting arithmetic was shown recently (see Flagg and Friedman 1986 for an elegant proof). Similar embeddings of Heyting arithmetic in appropriate modal extensions of Peano arithmetic contained in S4 Peano arithmetic can be proved with translations analogous to other modal translations we have considered. Can we prove such an embedding for the S4Grz extension of Peano arithmetic?

Besides the modal embeddings of Heyting's logic considered in this paper there is another famous type of embedding connected with Heyting's logic. Namely, classical logic can be embedded in Heyting's logic by various forms of the double-negation translation. Underlying this type of embedding there is also a modal translation.

Classical logic can be embedded by the translation s' into S5-like extensions of Heyting's logic, and in the case of propositional logic we can easily determine the minimal normal modal extension of H (where "normal" is understood relative to H) in which we can embed the classical propositional calculus C by s'. This is the system $H5p^-$, obtained by extending H with the modal postulates of Ks' and $\Box(\Box A \vee \Box\neg\Box A)$ (see Došen 1986). To determine the maximal normal extension of H in which we can embed C by s' is a straightforward matter (we have nothing like the complications connected with S4Grz). This is the system $C_{triv} = C + \Box A \leftrightarrow A$, obtained by extending H with $\Box(\Box A \vee \Box\neg\Box A)$ and $\Box A \leftrightarrow A$. This maximality of C_{triv} is proved like the fact that all consistent normal extensions of $H + \neg\Box\neg(A \rightarrow A)$ are included in C_{triv} (see Došen 1985, Lemma 1). The system C_{triv} is a conservative extension of C in L, but not of H in L. Can we find a maximal system (not necessarily unique) among the normal extensions of H in which C can be embedded by s', which are conservative extensions of H in L?

One such maximal conservative normal extension of H is the system $H_{dn} = H + \Box A \leftrightarrow \neg\neg A$. The embedding of C in H_{dn} by s' amounts to the simplest double-negation translation, where double negation is prefixed to every subformula. The translation s' is uneconomical for embedding C in H_{dn}: if $\Box$ in s'(A) is omitted in front of $\rightarrow$, $\wedge$ and $\neg$, we obtain a formula equivalent in H_{dn}. But the economy brought up by the translations t, t' and s is not now available. Of course, for embedding C in C_{triv} the economy can be total: all necessity operators are superfluous.

However, not all normal extensions of H, conservative with respect to H in L, in which we can embed C by s', are included in H_{dn}. One such extension which is not included in H_{dn} is obtained by adding $\Box A \rightarrow A$ to $H5p^-$. (That this system is conservative with respect to H in L may be proved with the help of models investigated in Ono 1977, Sotirov 1984, Došen 1985 and 1986a.) The economy brought up by the translations t, t' and s is now available, as well as a more thorough economy which omits every $\Box$ except those prefixed to propositional variables.

The general form of the embeddings considered here is the following. We have two nonmodal systems S' and S" such that S' is a proper subsystem of S", and we are able to show that:

(i) S' can be embedded by a modal translation in S" plus some modal postulates,

and vice versa:

(ii) S" can be embedded by a modal translation in S' plus some modal postulates.

Embeddings of H in modal systems with the nonmodal base C are of type (i), whereas embeddings of C in modal systems with the nonmodal base H are of type (ii). (The embeddings of classical and Heyting's logic into "linear logic" envisaged by Girard (1987) are like embeddings of type (ii).) For both types, one direction of our embeddings, that one which from the provability of A in the nonmodal system infers the provability of the translation of A in the modal system, is usually proved by a straightforward induction on the length of proof. The other direction is in principle more difficult to prove for type (i), because for type (ii) we usually have the following simple procedure. Our modal extension of S' must contain among other modal postulates the modal translations of theorems of S" missing from S'. To show that the provability of the modal translation of A in this extension of S' implies the provability of A in S", we use the fact that our modal extension of S' is included in S" plus $\Box A \leftrightarrow A$, and that this last system is a conservative extension of S". This simple procedure is not available for embeddings of type (i).

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